ON THE STRUCTURE OF HYPERCOMPLEX NUMBER SYSTEMS*

BY

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In this paper theorems regarding linear associative algebras are enunciated which are analogous to the JORDAN-HÖLDER theorems concerning the quotient groups and indices of composition in the theory of finite groups (cf. BURNSIDE, *Theory of Groups*, chapter 7); and of the corresponding VESSIOT-ENGEL theorems in the theory of continuous groups (Annales de l'École Normale Supériéure, vol. 9 (1892), p. 203; LIE-ENGEL, *Transformationsgruppen*, vol. 3, pp. 704, 765).

The methods used throughout are rational and hence the results apply to hypercomplex number systems in which the coefficients are restricted to be marks of a given field, finite or infinite.

§ 1.

We consider the linear associative algebra

$$E \equiv e_1 \cdots e_m e_{m+1} \cdots e_r e_{r+1} \cdots e_n, \tag{1}$$

where

$$e_{i_1} e_{i_2} = \sum_{i_3} \gamma_{i_1 i_2 i_3} e_{i_3} \qquad (i = 1, \cdots, n). \tag{2}$$

From the associativity property $(e_{i_1} e_{i_2}) e_{i_3} = e_{i_1}(e_{i_2} e_{i_3})$ and the linear independence of the units are deduced the relations

$$\sum_{i_3} (\gamma_{i_1 i_2 i_3} \gamma_{i_3 i_4 i_5} - \gamma_{i_1 i_3 i_5} \gamma_{i_2 i_4 i_3}) = 0 \quad (i_1, i_2, i_4, i_5 = 1, \cdots, n). \tag{3}$$

Addition of complexes. If $X_1, \cdots, X_m$ are any m independent numbers of the system E, all numbers linearly dependent on the $X_1, \cdots, X_m$ and only on these numbers are said to form a complex of order m. We may write $E_1 = X_1 \cdots X_m$. If, similarly, $E_2 = X'_1 \cdots X'_{m'}$, the complex formed of all members linearly dependent on $X_1, \cdots, X_m, X'_1, \cdots, X'_{m'}$ is called the sum of E_1 and E_2. The order of $E_1 + E_2$ is not greater than the sum of the orders of E_1 and E_2 but may be less; thus if $E_1 = e_1 \cdots e_m \cdots e_r$, $E_2 = e_{m+1} \cdots e_r \cdots e_{m'}$ the sum $E_1 + E_2$ is of order m'. According to this convention, $E + E_1 = E$, $E + E_2 = E$, $E + E = E$.

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* Presented to the Society (Chicago) December 29, 1904. Received for publication January 14, 1905.

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*Multiplication of complexes.** When E_1 and E_2 are defined as above, the product $E_1 E_2$ is defined as the complex of all numbers linearly dependent on $X_p X'_q$ $(p = 1, \cdots, m;\ q = 1, \cdots, m')$. It is easily shown from the definition of multiplication of complexes that the process is associative. Thus if we consider the system

	e_1	e_2
e_1	e_1	e_2
e_2	e_2	0

and let $E_1 \equiv e_1$, $E_2 \equiv e_2$, then $E_1 E_2 = e_1 e_2 = e_2$. It can easily be seen that if the main system E contains a modulus (say e_1) then $EE (= E^2) = E$; for, the modulus (unit) e_1 multiplied by $e_1 \cdots e_n$ on the right gives $e_1 \cdots e_n$ and therefore E^2 contains all the units of E. If E does not contain a modulus it may happen that in E^2 some of the units $e_1, \cdots, e_n$ do not appear and therefore $E^2 \lesseqgtr E$. The condition that E be a system, closed under multiplication, is $E^2 \lesseqgtr E$.

Semireducibility. The algebra E is said to be semireducible† when by a proper choice of units the following conditions are fulfilled:

$$\left.\begin{array}{ll} (4) & X \cdot {}_2X = {}_2X' \\ (5) & {}_2X \cdot X = {}_2X'' \end{array}\right\} \text{for every } X,\ {}_2X,$$

the general number of E being

$$X = {}_1X + {}_2X' = \sum_{j=1}^{m} x_j e_j + \sum_{k=m+1}^{n} x_k e_k .$$

It follows at once that the complex $E_2 = e_{m+1} \cdots e_n$ is closed under multiplication. The conditions (4) and (5) may be written

$$(4') \qquad E \cdot E_2 \lesseqgtr E_2,$$

$$(5') \qquad E_2 E \lesseqgtr E_2,$$

and, therefore, of course, $E_2 E_2 = E_2^2 \lesseqgtr E_2$. Under these conditions the subalgebra E_2 is said to be invariant in E.‡

THEOREM. *E_2 cannot contain the modulus of E among its numbers.* For, from (4) and (5) it is impossible that any number ${}_2I$ of E_2 shall have the property that ${}_2I \cdot X = X \cdot {}_2I = X$ for every number X of E.

Accompanying system,§ *complementary system.* If

$$E = E_1 + E_2 \equiv e_1 \cdots e_m e_{m+1} \cdots e_n,$$

* FROBENIUS, Berliner Sitzungsberichte, 1895, p. 164.

† Transactions, vol. 4 (1903), p. 440.

‡ CARTAN, Annales de Toulouse, vol. 12 (1898).

§ MOLIEN's *Begleitendes System*, Mathematische Annalen, vol. 41 (1893), p. 93.

we may write, in well known notation, $E = E_1 (\text{mod } E_2)$, $E = E_2 (\text{mod } E_1)$; similarly, if $X = {}_1X + {}_2X$, where X belongs to E, ${}_1X$ to E_1, ${}_2X$ to E_2, we may write $X = {}_1X (\text{mod } E_2)$ or $X = {}_2X (\text{mod } E_1)$. Regarding as equal * any two numbers of E which are equal, modulo E_2, we get, when E_2 is an invariant subsystem, a hypercomplex number system K which is said to *accompany* E and to be *complementary to the invariant subsystem* E_2 *with respect to* E. It is evidently sufficient to show that the system so defined is associative. If ${}_1X_\alpha$, ${}_1X_\beta$, ${}_1X_\gamma$ are three numbers of E_1 and if

$$ {}_1X_\alpha \cdot {}_1X_\beta = X_{\alpha\beta} = {}_1X_{\alpha\beta} + {}_2X_{\alpha\beta}, \qquad {}_1X_\beta \cdot {}_1X_\gamma = X_{\beta\gamma} = {}_1X_{\beta\gamma} + {}_2X_{\beta\gamma}, $$

then

$$ ({}_1X_\alpha \cdot {}_1X_\beta) \cdot {}_1X_\gamma = {}_1X_{\alpha\beta} \cdot {}_1X_\gamma + {}_2X_{\alpha\beta} \cdot {}_1X_\gamma = {}_1X_{(\alpha\beta)\gamma} + {}_2X' $$

if $E_2 E_1 \leqq E_2$. Similarly

$$ {}_1X_\alpha \cdot ({}_1X_\beta \cdot {}_1X_\gamma) = {}_1X_{\alpha(\beta\gamma)} + {}_2X''. $$

Now by the associative law and the linear independence of the units we have

$$ {}_1X_{(\alpha\beta)\gamma} \equiv {}_1X_{\alpha(\beta\gamma)}, $$

and therefore

$$ ({}_1X_\alpha \cdot {}_1X_\beta) \cdot {}_1X_\gamma \equiv {}_1X_\alpha \cdot ({}_1X_\beta \cdot {}_1X_\gamma) (\text{mod } E_2). $$

The above may also be deduced from a consideration of the $\gamma_{i_1 i_2 i_3}$'s. Using (3) it is seen that

$$ (3_j) \qquad \sum_{i_3=1}^{n} (\gamma_{j_1 j_2 i_3} \gamma_{i_3 j_4 j_5} - \gamma_{j_1 i_3 j_5} \gamma_{j_2 j_4 i_3}) = 0 \qquad (j_1, j_2, j_4, j_5 = 1, \cdots, m). $$

Now if

$$ E_2 E_1 \leqq E_2, \qquad E_1 E_2 \leqq E_2, $$

or,

$$ {}_2X \cdot {}_1X = {}_2X', \qquad {}_1X \cdot {}_2X = {}_2X'', $$

that is to say, if every $\gamma_{kjj} = 0$ and every $\gamma_{jkj} = 0$, we may write (3_j) in the form

$$ (3_j') \qquad \sum_{j_3=1}^{m} (\gamma_{j_1 j_2 j_3} \gamma_{j_3 j_4 j_5} - \gamma_{j_1 j_3 j_5} \gamma_{j_2 j_4 j_3}) = 0.\dagger $$

§ 2.

By $E_1 \frown E_2$ is meant the common complex of E_1 and E_2. Thus if in (1), $E_1 \equiv e_1 \cdots e_m \cdots e_r$, $E_2 \equiv e_{m+1} \cdots e_r \cdots e_n$, then $E_1 \frown E_2 \equiv e_{m+1} \cdots e_r$. $E_1 \frown E_2 = 0$ states that E_1 and E_2 have no numbers in common.

* FROBENIUS, l. c., p. 634.

† According to MOLIEN, loc. cit., p. 92, it is necessary that $EE_2 < E_2$, $E_2 E \leqq E_2$. Evidently, from the text, we can have an accompanying system with the milder conditions $E_2 E_1 \leqq E_2$, $E_1 E_2 \leqq E_2$.

It may be remarked at this point that the γ_{jjj} are the structure constants of K.

Reducibility. According to PEIRCE* an algebra E is said to be mixed (or reducible) if new units can be introduced such that $E = E_1 + E_2$, where $E_1^2 \leqq E_1$, $E_2^2 \leqq E_2$, $E_1 E_2 \leq E_1 \frown E_2$, $E_2 E_1 \leqq E \frown E_2$.†

THEOREM I. *If E_1 is a maximal invariant subalgebra of E and if there exists a second invariant subalgebra E_1' of E, then either E is reducible or else E_1' is a subalgebra E_1.*

Since E_1 and E_1' are invariant we have (*a*) $EE_1 \leqq E_1$, (*b*) $E_1 E_1' \leqq E_1$, (*c*) $EE_1' \leqq E_1'$, (*d*) $E_1' E \leqq E_1'$. We have now that $E_1 + E_1'$ is an invariant subalgebra of E for $E(E_1 + E_1') \leqq E_1 + E_1'$, $(E_1 + E_1')E \leqq E_1 + E_1'$ and since E_1 is is maximal by hypothesis $E_1 + E_1' = E$ or else $E_1 + E_1' = E_1$. If $E_1 + E_1' = E$ the algebra is easily seen to be reducible. First, $E_1^2 \leqq E_1$ by (*a*), $E_1'^2 \leqq E_1'$ by (*c*), $E_1 E_1' \leqq E_1$ by (*b*) and $E_1 E_1' \leqq E_1'$ by (*c*) and therefore $E_1 E_1' \leqq E_1 \frown E_1'$. Similarly by (*b*) and (*d*), $E_1' E_1 = E_1 \frown E_1'$. In the second case, where $E_1 + E_1' = E_1$, it is evident that E_1' is a subalgebra of E_1.

THEOREM II. *If E_1 and E_1' are maximal invariant subalgebras of E, and if $E_1 \frown E_1' = F \neq 0$, then F is a maximal invariant subalgebra of both E_1 and E_1'.*

By theorem I, $E = E_1 + E_1'$. Let

$$E = E_0 + E_1 \qquad (E_0 \frown E_1 = 0),$$
$$= E_0' + E_1' \qquad (E_0' \frown E_1' = 0).$$

Evidently then $E = E_0 + E_0' + F$. Because the products $E_1 F$, FE_1, $E_1' F$, FE_1' must be contained in E_1 and in E_1' it follows that

$$E_1 F \leqq F, \qquad FE_1 \leqq F, \qquad E_1' F \leqq F, \qquad FE_1' \leqq F,$$

and therefore that F is invariant in both E_1 and E_1'.

Suppose now that F is not maximal in E_1 and let $G + F$ (where $G \frown F = 0$) be a maximal invariant subsystem of E_1, i. e.,

$$E_1(G + F) \leqq G + F, \qquad (G + F)E_1 \leqq G + F.$$

Consider the complex $G + E_1'$. Clearly $G \frown E_1' = 0$ and

$$GE_1' \leqq F, \qquad E_1' G \leqq F,$$

since the products must both lie in E_1 and E_1'. Therefore, since

$$E_1 G \leqq G + F, \qquad E_1 E_1' \leqq F, \qquad E_1' G \leqq F, \qquad E_1'^2 \leqq E_1',$$

we have

$$E(G + E_1') = (E_1 + E_1')(G + E_1') \leqq G = E_1'.$$

It is shown in a similar manner that $(G + E_1')E \leqq G + E_1'$. Thus, if F is not a maximal invariant subalgebra of E_1, E_1' will not be a maximal invariant

* American Journal, vol. 4 (1881), p. 100.

† In SCHEFFERS's definition, Mathematische Annalen, vol. 39 (1891), p. 317, $E_1 \frown E_2 = 0$.

subalgebra of E (since $G + E'_1$ has been proved invariant and $G \frown E'_1 = 0$), which is contrary to the hypothesis. It is shown in exactly the same way that F is maximal in E'_1.

THEOREM III. *Let $E, E_1, E_2, \cdots$ be a normal series of subalgebras of E (i. e., E_r is a maximal invariant subalgebra of E_{r-1}, $E_0 = E$) and let $K_1, K_2, \cdots$ be a series of complementary algebras, such that K_r accompanies E_{r-1} and is complementary to E_r. Under these assumptions the series $K_1, K_2, \cdots$ is, apart from the order, independent of the choice of the series $E, E_1, E_2, \cdots$. In other words, if $E, E'_1, E'_2, \cdots$ is any other normal series, the complementary series of algebras $K'_1, K'_2, \cdots$ which it defines is the same as the series $K_1, K_2, \cdots$, apart from the sequence.*

According to the demonstration of theorem 1, we have (if $E_1 \neq E'_1$)

$$E = E_1 + E'_1. \tag{6}$$

Let $E_1 \frown E'_1 = F_1$, then

$$E_1 = F_1 + D_1 \qquad (F_1 \frown D_1 = 0), \tag{7}$$

$$E'_1 = F_1 + D'_1 \qquad (F_1 \frown D'_1 = 0),$$

and it is evident that

$$E = F_1 + D_1 + D'_1. \tag{8}$$

By theorem 2, F_1 is a maximal invariant subalgebra of E_1 and E'_1.

If F_1, F_2, F_3 is a normal series of F_1, then $E, E_1, F_1, F_2, \cdots$ and $E, E'_1, F_1, F_2, \cdots$ are two new normal series of E. We will now have to prove that the proposition is true for these two series. For this purpose it is merely necessary to prove that the complementary algebras defined by E, E_1, F_1 and E, E'_1, F_1 are the same. If

$$E \equiv e_1 \cdots e_{d_1} e_{d_1+1} \cdots e_t e_{t+1} \cdots e_n,$$

$$E_1 \equiv e_1 \cdots e_{d_1} e_{d_1+1} \cdots e_t,$$

$$E'_1 \equiv e_{d_1+1} \cdots e_t e_{t+1} \cdots e_n,$$

then

$$E_1 \frown E'_1 = F_1 \equiv e_{d_1+1} \cdots e_t,$$

$$D_1 \equiv e_1 \cdots e_{d_1}, \qquad D'_1 = e_{t+1} \cdots e_n.$$

Now the complementary algebra of $E_1 = F_1 + D_1$ with respect to E is defined by the $(n-t)^3$ characteristic constants $\gamma_{p_1 p_2 p_3}$ where

$$\sum_{p_3} (\gamma_{p_1 p_2 p_3} \gamma_{p_3 p_4 p_5} - \gamma_{p_1 p_3 p_5} \gamma_{p_2 p_4 p_3}) = 0 \quad (p = t+1, t+2, \cdots, n). \tag{9}$$

Furthermore, the complementary algebra of F_1 with respect to E_1 ($E_1 = F_1 + D_1$) is defined by the d_1^3 characteristic constants $\gamma_{q_1 q_2 q_3}$ where

$$\sum_{q_3} (\gamma_{q_1 q_2 q_3} \gamma_{q_3 q_4 q_5} - \gamma_{q_1 q_3 q_5} \gamma_{q_2 q_4 q_3}) = 0 \qquad (q = 1, \cdots, d_1). \tag{10}$$

Now it is seen in the same manner that the complementary algebra of E'_1 with respect to E is defined by (10) and the complementary algebra of F_1 with repect to E'_1 is defined by (9), and therefore the normal complementary series defined by E, E_1, F_1 is identical, apart from the sequence, with the normal complementary series defined by E, E'_1, F_1.

It is now necessary to show that the normal series E, E_1, $E_2 \cdots$, E, E_1, F_1, F_2, $\cdots$ define the same complementary series, and similarly for E, E'_1, E'_2, $\cdots$, E, E'_1, F_1, F_2, $\cdots$. This amounts to proving the theorem for the algebras E_1 and E'_1, which involves a finite number of repetitions of the above proof.

§ 3.

Let E, E_1, E_2, $\cdots$ be a normal series of E, let a_s be the order (number of units) of E_s and let l_s be the difference between a_{s-1} and the maximal order of a subalgebra * of E_{s-1} which contains $E_s (E_0 = E)$.

THEOREM IV. *The numbers* l_1, l_2, $\cdots$, *apart from their sequence, are independent of the choice of the normal series.*

This theorem is an immediate consequence of theorem 3, the integer l_s being regarded as the difference between the order of the complementary algebra K_s of E_s with respect to E_{s-1} and the order of a maximal subalgebra of $K_s (s = 1, 2, \cdots)$.

Similarly it follows also *that the series of integers* $k_s = a_{s-1} - a_s$ *is independent of the choice of the normal series.*

Simple algebras. An algebra which contains no invariant subalgebra is said to be simple.

THEOREM V. *The complementary algebras* K_1, K_2, $\cdots$ *defined by the normal series* E, E_1, $E_2 \cdots$ *are all simple.*†

In order to prove this, consider the algebra $A = B + A_1$ (where $B \frown A_1 = 0$) wherein A_1 is a maximal invariant subalgebra, and let K be the complementary system of A_1 with respect to A. Suppose now that K has an invariant subalgebra K_1, $K = K' + K_1$ (where $K' \frown K_1 = 0$). This says that

$$(11) \qquad KK_1 \leqq K_1, \qquad K_1 K \leqq K_1.$$

From the correspondence between K and B‡ it is seen that we can write $B = B' + B_1$ (where $B' \frown B_1 = 0$) where, by means of (11),

$$(12) \qquad BB_1 \leqq B_1 + A_1, \qquad B_1 B \leqq B_1 + A_1.$$

Consider now the complex $B_1 + A_1$. We have, since A_1 is invariant in A, and by means of (12),

* i. e., the order of the subalgebra containing the largest number of independent units.

† This theorem is implicitly contained in MOLIEN's paper, loc. cit., p. 96.

‡ The γ_{ijj} of K are obtained by writing every $\gamma_{ijk} = 0$ in the γ_{iji} of B.

$$A(B_1 + A_1) = (B + A_1)(B_1 + A_1) \leqq B_1 + A_1;$$

and similarly

$$(B_1 + A_1)A \leqq B + A_1.$$

Thus $B_1 + A_1$ is an invariant subalgebra of A and since $B_1 \frown A_1 = 0$, it follows that $B_1 + A_1 > A_1$ which is impossible since A_1 is maximal by hypothesis. Therefore K is simple.

The chief series of an algebra. The series E, P_1, P_2, $\cdots$ is said to be a chief or principal series (chief composition series) when P_s is a maximal subalgebra of P_{s-1} which is invariant in $E(P_0 = E)$. For the chief series it is shown in exactly the same manner as for the normal series that *the system of complementary algebras C_1, C_2, C_3, $\cdots$ is invariant, apart from the sequence (C_s being complementary to P_s with respect to P_{s-1}).*

We can now define a series of chief indices of composition analogous to the indices of composition above, and show that for a given algebra E, *the system of chief indices of composition is independent of the choice of the chief series, apart from the sequence.*

THE UNIVERSITY OF CHICAGO,
December, 1904.